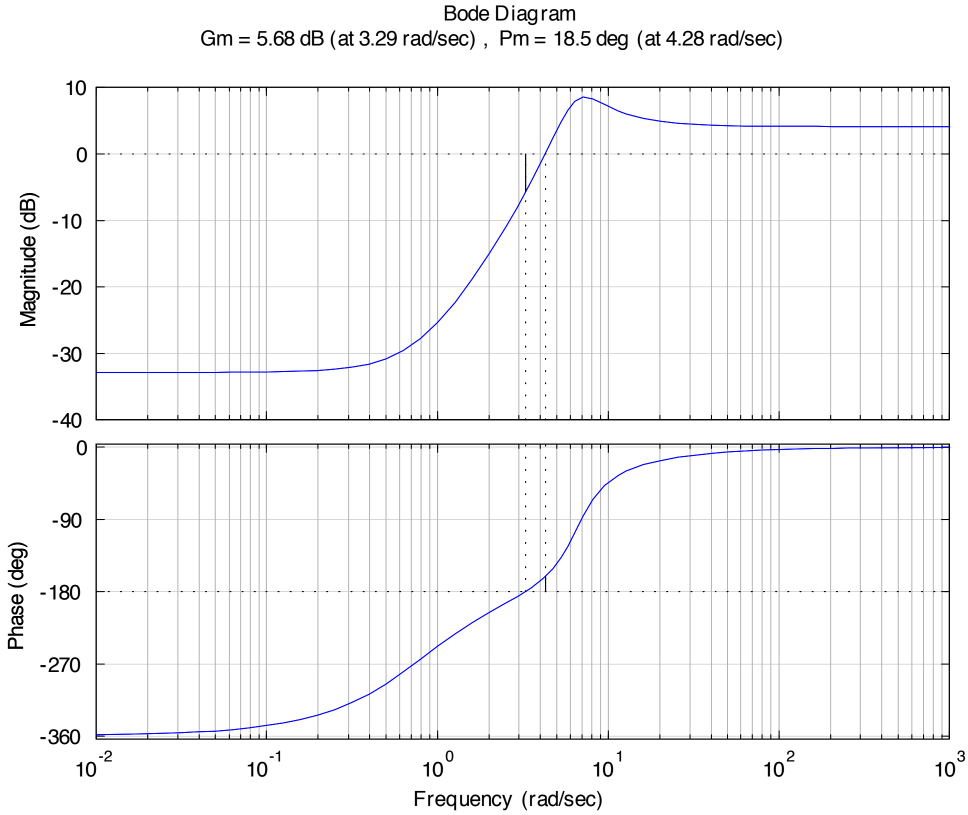


Problem 7.49: Bode plot of the compensated system.

50. Suppose a DC drive motor with motor current u is connected to the wheels of a cart in order to control the movement of an inverted pendulum mounted on the cart. The linearized and normalized equations of motion corresponding to this system can be put in the form

$$\ddot{\theta} = \theta + v + u,$$
$$\dot{v} = \theta - v - u,$$

where,

$$\theta = \text{angle of the pendulum},$$
$$v = \text{velocity of the cart}.$$

a) We wish to control θ by feedback to u of the form,

$$u = -K_1\theta - K_2\dot{\theta} - K_3 v.$$

Find the feedback gains so that the resulting closed-loop poles are located at -1, $-1 \pm j\sqrt{3}$.
b) Assume that θ and v are measured. Construct an estimator for θ and $\dot{\theta}$ of the form,

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{L}(y - \hat{y}),$$

where $\mathbf{x} = [\ \theta \quad \dot{\theta}\ ]^T$ and $y = \theta$. Treat both v and u as known. Select $\mathbf{L}$ so that the estimator poles are at -2 and -2.
c) Give the transfer function of the controller, and draw the Bode plot of the closed-loop system, indicating the corresponding gain and phase margins.
d) Using MATLAB, plot the response of the system to an initial condition on θ, and give a physical explanation for the initial motion of the cart.

Solution:

(a) Defining the state $\mathbf{x} = [\theta\ v]^T$, the system is written as,

$$\begin{aligned} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{v} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u, \\ \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u. \end{aligned}$$

Using $\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = 0$ with $\mathbf{K} = [\ k_1 \quad k_2 \quad k_3\ ]$, we find the characteristic equation,

$$s^3 + s^2(1 - k_3 + k_2) + s(k_1 - 1) + 2(k_3 - 1) = 0.$$

The desired characteristic equation is,

$$(s+1)((s+1)^2 + 3) = s^3 + 3s^2 + 6s + 4 = 0.$$

Comparing coefficients, $\mathbf{K} = [\ 7 \quad 5 \quad 3\ ]$. This result can be verified using the MATLAB place command.

(b) The estimator equations (both explicitly and symbolically) for estimating $\hat{\mathbf{x}} = [\ \theta \quad \dot{\theta}\ ]^T$ are,

$$\begin{aligned} \begin{bmatrix} \hat{\dot{\theta}} \\ \hat{\ddot{\theta}} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\theta} \\ \hat{\dot{\theta}} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \mathbf{L}(y - \hat{y}), \\ &= \mathbf{A}_e \widehat{\mathbf{x}} + \mathbf{B}_v v + \mathbf{B}_u u + \mathbf{L}(y - \hat{y}). \end{aligned}$$

where u and v are assumed to be known. The output equations for the plant and the estimator are,

$$\begin{aligned} y &= \mathbf{C}\mathbf{x} = [\ 1 \quad 0 \quad 0\ ]\,\mathbf{x}, \\ \hat{y} &= \mathbf{C}_e \hat{\mathbf{x}} = [\ 1 \quad 0\ ]\,\hat{\mathbf{x}}. \end{aligned}$$

With $\mathbf{L} = [\ l_1 \quad l_2\ ]^T$, the characteristic equation becomes,

$$\det(s\mathbf{I} - \mathbf{A}_e + \mathbf{L}\mathbf{C}_e) = s^2 + sl_1 + l_2 - 1 = 0.$$

Equating with the desired characteristic equation,

$$(s+2)(s+2) = s^2 + 4s + 4,$$

we have $\mathbf{L} = [\ 4 \quad 5\ ]^T$. This result can be verified using the MATLAB place command.

(c) Construct the feedback u in terms of both the measured signal v and the estimated state $\hat{\mathbf{x}}$. Using the feedback gains from (a), we have,

$$\begin{aligned} u &= -K_1\hat{\theta} - K_2\dot{\hat{\theta}} - K_3 v, \\ &= -\mathbf{K}_e\hat{\mathbf{x}} - K_3 v. \end{aligned}$$

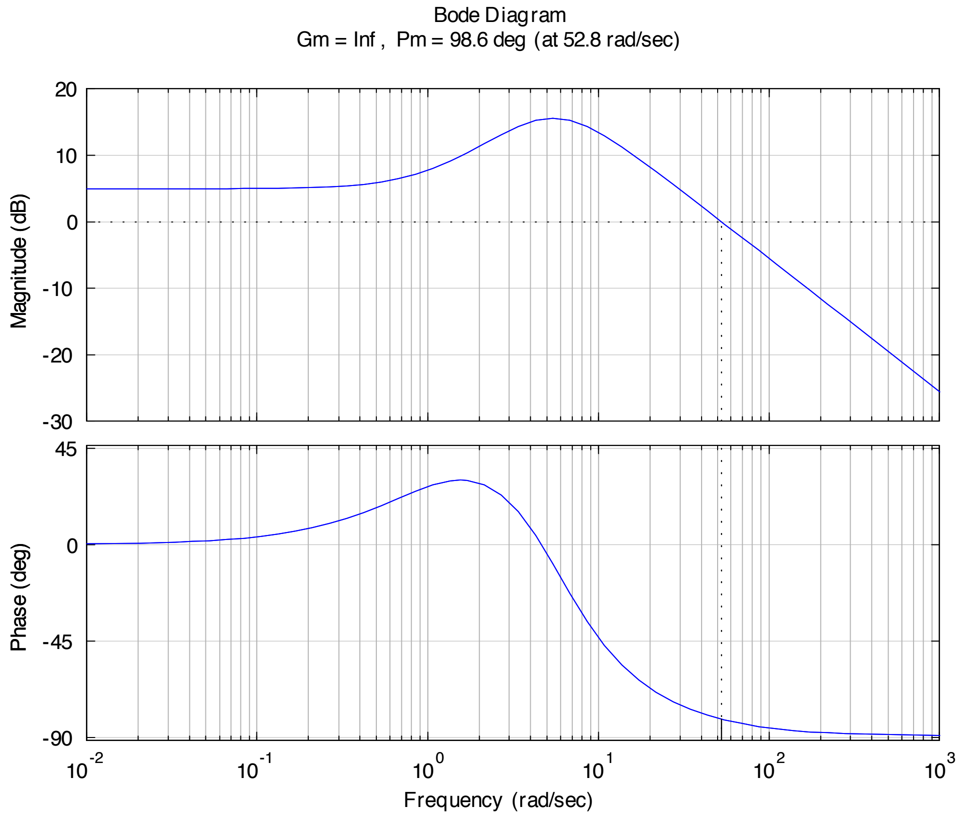
Plugging this expression for u into the estimator equation we have,

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= (\mathbf{A}_e - \mathbf{B}_u\mathbf{K}_e - \mathbf{L}\mathbf{C}_e)\hat{\mathbf{x}} + (\mathbf{B}_v - \mathbf{B}_u K_3)v + \mathbf{L}y, \\ u &= -\mathbf{K}_e\hat{\mathbf{x}} - K_3 v. \end{aligned}$$

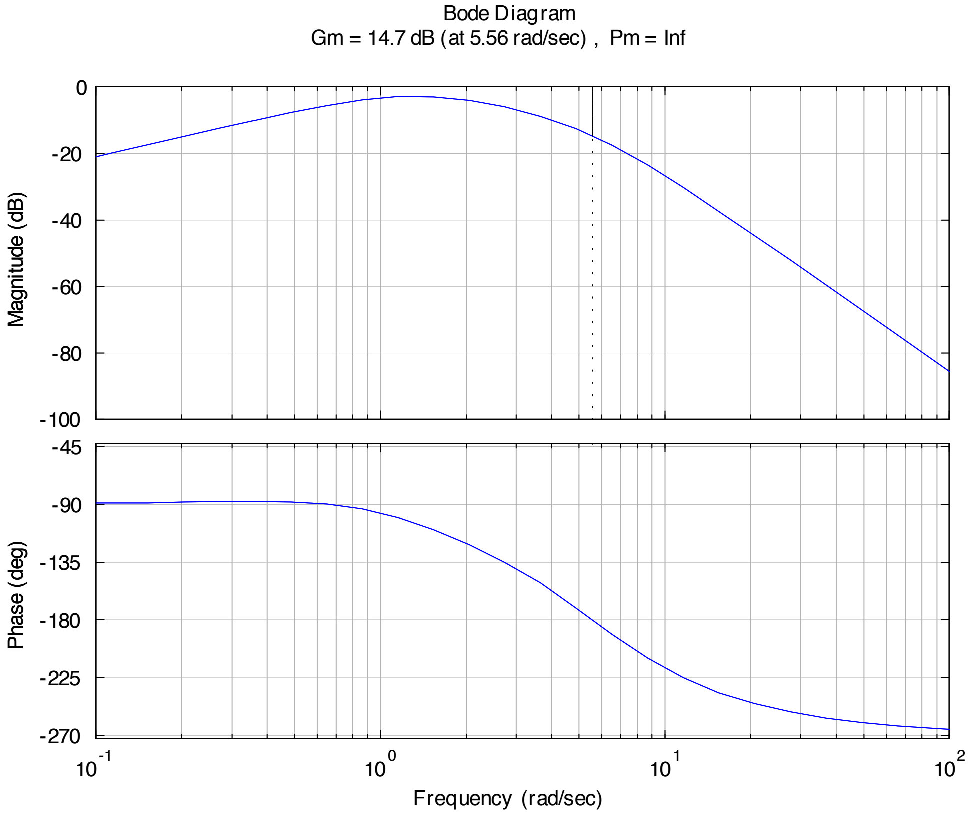
The transfer function from y to u can now be read directly from these two equations by setting all of the auxiliary inputs to zero, i.e., $v = 0$. Thus,

$$D_c(s) = -\mathbf{K}_e(s\mathbf{I} - \mathbf{A}_e + \mathbf{B}_u\mathbf{K}_e + \mathbf{L}\mathbf{C}_e)^{-1}\mathbf{L} = \frac{-(53s+55)}{s^2+9s+31}.$$

The Bode plots are shown next.



Open-loop Bode plot of compensator transfer function for Problem 7.50.

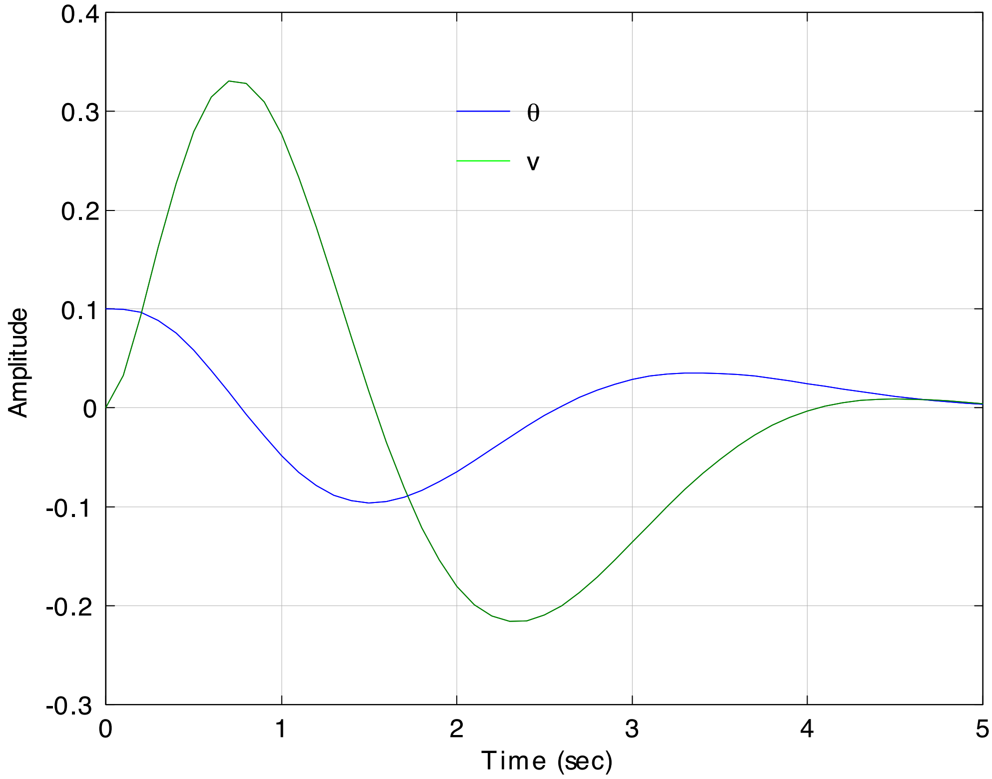


Bode plot of the compensator and plant together for Problem 7.50.

(d) One approach to simulating the system is to augment the plant and estimator equations into one matrix. Recognizing that $v = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \mathbf{C}_v \mathbf{x}$, we can eliminate u and v.

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u = (\mathbf{A} - \mathbf{B}K_3\mathbf{C}_v)\mathbf{x} - \mathbf{B}\mathbf{K}_e\hat{\mathbf{x}} \\
\dot{\hat{\mathbf{x}}} &= \mathbf{A}_e\hat{\mathbf{x}} + \mathbf{B}_v v + \mathbf{B}_u u + \mathbf{L}(y - \hat{y}) \\
&= (\mathbf{B}_v\mathbf{C}_v - \mathbf{B}_u K_3 \mathbf{C}_v + \mathbf{L}\mathbf{C})\mathbf{x} + (\mathbf{A}_e - \mathbf{L}\mathbf{C}_e - \mathbf{B}_u\mathbf{K}_e)\hat{\mathbf{x}}.
\end{aligned}$$

This is now easily implemented using the MATLAB command `lsim`. The figure on the next page shows the closed-loop system response due to an initial angle of $\theta = 0.1$ rad with respect to a vertical line. The initial motion of the cart is in the direction that the pendulum is leaning (due to the initial condition). Physically, if the cart moved away from the direction that the pendulum was leaning, then it would cause the angle to increase eventually toppling the pendulum.



Angle of pendulum and the velocity of the cart given an initial angle for Problem 7.50.

51. Consider the control of

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{s(s+1)}.$$

a) Let $y = x_1$ and $\dot{x}_1 = x_2$, and write state equations for the system.
b) Find K_1 and K_2 so that $u = -K_1 x_1 - K_2 x_2$ yields closed-loop poles with a natural frequency $\omega_n = 3$ and a damping ratio $\zeta = 0.5$.
c) Design a state estimator for the system that yields estimator error poles with $\omega_{n1} = 15$ and $\zeta_1 = 0.5$.
d) What is the transfer function of the controller obtained by combining parts (a) through (c)?
e) Sketch the root locus of the resulting closed-loop system as plant gain (nominally 10) is varied.

Solution:

The state equations are,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}. \end{aligned}$$

(b) K = place(A, B, roots([1 2 ∗ zeta ∗ wn wn^2]))= [0.9 0.2].